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Potentials and T-Duality

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ABSTRACT

We discuss the application of T-duality to massive supersymmetric sigma models. In particular $(1,1)$ supersymmetric models with off-shell central charges reveal an interesting structure. The T-duality transformations of the BPS states of these theories are also discussed and an explicit example of Q-kinks is given.

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1. Introduction

T-duality is a central tool in the the study of superstring vacua. While T-duality of massless sigma models has been studied at length the, T-duality of massive sigma models has received little attention. This is perhaps because the presence of the potential terms violates conformal invariance, thus obscuring the connection with perturbative string theory and conformal field theory. However, at the level of the sigma model action, T-duality is simply a Gaussian integral over a Killing direction and thus the duality may in this sense be applied to a massive sigma model. Recent advances in non-perturbative string theory have shown that in the presence of D-branes, string worldsheets do possess massive terms [2]; one may anticipate that the complete role of worldsheet potentials has yet to be realised. It is therefore of some interest to understand the implications of T-duality for such theories. Much of the discussion is then readily extended to other p-brane actions with potentials.

Consider the following bosonic sigma model action

$$S = \frac{1}{2\pi\alpha'} \int d^2\sigma \ \Big\{ (g_{IJ}\eta^{\mu\nu} + b_{IJ}\epsilon^{\mu\nu})\partial_\mu X^I \partial_\nu X^J - V(X) \Big\} \ , \tag{1.1}$$

where $I, J = 0, 1, 2, ..., (D-1)$ and V is a potential term. Let us suppose that the target space has a $U(1)$ isometry associated with the direction X^0, with the other coordinates labelled by X^i. In this simple bosonic model the potential transforms trivially under the duality, since it is independent of X^0. Furthermore, if we view (1.1) as the bosonic sector of a supersymmetric theory then we again find that the potential transforms trivially by using $(1,1)$ superfields. However, it is precisely in the cases for which the target space possesses a Killing vector $k = \partial/\partial X^0$ that there is another form of the potential compatible with $(1,1)$ supersymmetry [4,3]. This potential can not be written using (standard) $(1,1)$ superfields because of the appearance of off-shell central charges, and instead must be defined as the length

of the Killing vector k

$$V = \frac{1}{4}m^2 g_{IJ}k^I k^J = \frac{1}{4}m^2 g_{00} \ , \tag{1.2}$$

where m is an arbitrary mass parameter. For the simplest background with $g_{0i} = b_{0i} = 0$, duality inverts the metric component g_{00}. Naively following the usual description of T-duality for the massless sigma model, the potential for the dual background in such a case would be V^{-1}; hence it would appear that supersymmetry could be spontaneously broken in the dual theory! A related problem is that these theories can possess BPS solitons interpolating between the different vacua of the potential and these must always exist in the spectrum. The main motivation of this paper is to resolve these issues and examine the behaviour of the BPS states under T-duality. We will see that the potential is always invariant under T-duality and that the BPS multiplets are also preserved. However, the dual potential has a different form in superspace and correspondingly different BPS solutions.

2. The dual potential

We initially consider supersymmetric sigma-models with potentials defined by the Killing vector $k = \partial/\partial X^0$. In such a case we may not express the action in terms of $(1,1)$ superfields since the algebra possesses off-shell central charges [3], and we lose manifest supersymmetry. In terms of $(1,0)$ superfields the action is

$$S = -i\int d^2\sigma d\theta^+ \ (g_{IJ} + b_{IJ})D_+\Phi^I \partial_= \Phi^J + ig_{IJ}\Psi_-^I \nabla_+^{(+)}\Psi^J + ims_I\Psi^I \ . \tag{2.1}$$

Here $\Phi^I = X^I + \theta^+\psi_+^I$, $\Psi^I = \psi_-^I - \theta^+ F^I$ and $\nabla^{(+)}$ is the covariant derivative with torsion. The action (2.1) has manifest $(1,0)$ supersymmetry for any choice of

the co-vector field s_I.* Elimination of the auxiliary field by its equation of motion $2g_{IJ}F^J - ms_I = 0$ leads to the potential $V = \frac{1}{4}m^2 g^{IJ}s_I s_J$.

Adapting the results of [5] to $(1,0)$ supersymmetry we find the relation between the dual and original superfields to be

$$\begin{aligned} D_+\hat{\Phi}^0 &= -g_{00}D_+\Phi^0 - (g_{0i}+b_{0i})D_+\Phi^i \\ D_+\hat{\Psi}^0_- &= -g_{00}D_+\Psi^0_- - (g_{0i}+b_{0i})D_+\Psi^i_- \\ \partial_=\hat{\Phi}^0 &= +g_{00}\partial_=\Phi^0 + (g_{0i}-b_{0i})\partial_=\Phi^i \\ \hat{\Phi}^i &= \Phi^i \\ \hat{\Psi}^i_- &= \Psi^i_- \ . \end{aligned} \tag{2.2}$$

To dualise in such a way as to preserve manifest $(1,0)$ supersymmetry we must remove the dual auxiliary field by it's equation of motion in the dual theory. Equation (2.2) implies that

$$\hat{F}^0 = -g_{00}F^0 - (g_{0i}+b_{0i})F^i\,, \qquad \hat{F}^i = F^i \ . \tag{2.3}$$

Now we note that this transformation is consistent with the equations of motion in the dual model if and only if

$$\hat{s}_0 = -\frac{s_0}{g_{00}}\,, \qquad \hat{s}_i = s_i - (\frac{g_{0i}}{g_{00}} + \frac{b_{0i}}{g_{00}})s_0 \ . \tag{2.4}$$

However, one can easily check that $\hat{V} = \hat{g}_{IJ}\hat{F}^I\hat{F}^J = g_{IJ}F^IF^J = V$. Thus the potential is invariant under T-duality even though the co-vector is not. The supersymmetry is preserved and the vacua of the dual theory are precisely the same as those in the original theory. We can also see from (2.4) that if $s_0 = 0$, as is the case for models possessing a $(1,1)$ superspace form, then both the potential and s_I are invariant under T-duality.

⋆ There is a much more general class of $(1,0)$ supersymmetric sigma models than (2.1) where Ψ_- and s lie in an arbitrary vector bundle over the target space [3]. It is not hard to see from the discussion below that these fields, which include the mass terms, transform trivially under T-duality unless the vector bundle is identified with the tangent bundle.

We now consider the case when the action (2.1) admits $(1,1)$ supersymmetry. The co-vector field must take the form $s_I = k_I - u_I$, where $\partial_{[J}u_{K]} = k^I H_{IJK}$ and $k^I u_I = 0$ [3]. For our choice of coordinates we find that

$$s_0 = g_{00}\,, \qquad s_i = g_{0i} + (b_{0i} - \partial_i\lambda)\ , \tag{2.5}$$

where λ is an arbitrary function of X^i, which may be absorbed into b_{0i}, yielding the potential

$$V = \frac{1}{4}m^2\left(g_{00} + g^{ij}b_{0i}b_{0j}\right)\ . \tag{2.6}$$

To dualise this model we apply the transformation (2.4) to obtain the dual form of the co-vector field

$$\hat{s}_0 = -1\,, \qquad \hat{s}_i = 0\ . \tag{2.7}$$

It follows as before that $\hat{V} = V$. It is easy to see that this new form also possess $(1,1)$ supersymmetry, only this time without off-shell central charges. The obstruction to a full superspace expression is the fact that the superspace potential $\hat{\Lambda} = \hat{X}^0$ is only locally defined in the target space, unlike the globally defined derivative terms which appear in the equations of motion. Thus there are effectively two classes of $(1,1)$ supersymmetric models which do not possess a $(1,1)$ superspace form: those defined in terms of Killing vectors, and those defined by topologically non-trivial closed forms. T-duality exchanges these two classes and so in a sense exchanges the isometry for the cohomology.

An important feature of the T-duality of a massless sigma model is that if the β-functions of the original model vanish, then so do the β-functions of the dual model (they are in fact "covariant" under T-duality [1]). A natural question to ask is whether or not the β-functions of the dual massive model are invariant under duality, and also to investigate the effects of the massive terms on the beta functions. One can easily see from dimensional analysis that the metric and anti-symmetric tensor β-functions of a massive sigma model are unaffected by the mass

terms. Thus these must vanish in the dual model if they vanish in the original one. So let us just consider the massive β-function here. Following the result of [7] and combining it with the one loop divergence found in [8], whose notation we adopt, we find that the contribution to the trace anomaly is

$$\beta_m = m \int d^2 x d\theta^+ \left\{ s_I - \frac{1}{2}\alpha' \left(\nabla^{(-)2} s_I - 2\partial^K \phi \nabla_K^{(-)} s_I \right) \right\} \Psi_-^I \ , \qquad (2.8)$$

where ϕ is the dilaton, which enters through the Fradkin-Tseytlin term in the action. This integral consists of a classical contribution, caused by the addition of the potential term to the conformally invariant massless sigma model, and the order α' quantum corrections. We wish to investigate in which situations the order α' piece vanishes, and also whether or not the form of the integral is invariant under the T-duality procedure. In order to proceed, we restrict ourselves to the bosonic sector, for which $\Psi_-^I = \frac{1}{2} m\theta^+ s^I$. The classical contribution to (2.8)is simply the potential V, which is invariant under T-duality. The order α' term is less simple: in general it is neither zero, nor invariant under T-duality. In addition, contrasting the the massless case, the potential β-function is not covariant under T-duality. There are, however, certain special classes of cases for which we can say something more constructive: If $b_{0i} = 0$ then we find that both the original and dual expressions vanish, and the anomaly is entirely classical. Note that according to the torsion, the b_{IJ} field is only defined up to a total derivative; specifying b_{0i} to be zero is equivalent to making a particular choice for λ in the definition (2.5), corresponding to the case for which the potential is given by (1.2). This includes the cases of hyper-Kähler string backgrounds such as the Q-kink soliton sigma model, discussed later. Similar requirements on λ have been noted previously in the context of one loop finiteness [8]. Finally, since $b_{0i} \leftrightarrow g_{0i}$ under T-duality, $(1,0)$ the previous comments imply that the anomaly also vanishes for any sigma model with s_I of the form (2.7)with vanishing metric cross terms g_{i0}.

3. BPS states and an example

Although the potential of the dual theory is invariant under T-duality, the superspace form of the theory changes under the transformation. Therefore the BPS states and Bogomoln'yi equations of the model must also change. To examine this it is sufficient just to consider the bosonic sector of (2.1). Using the standard procedure we express the energy as a sum of squares plus a topological term

$$\begin{aligned} E &= \int dx\, g_{IJ} \left(\dot{X}^I \dot{X}^J + X^{'I} X^{'J} + \frac{1}{4} m^2 s^I s^J \right) \\ &= \int dx \left(g_{IJ} (\dot{X}^I - \frac{1}{2} m k^I)(\dot{X}^J - \frac{1}{2} m k^J) + g_{IJ} (X^{'I} - \frac{1}{2} m u^I)(X^{'J} - \frac{1}{2} m u^J) \right) \\ &\quad + \mathcal{T} \ . \end{aligned} \tag{3.1}$$

In (3.1) we have introduced a transparent notation and the 'topological' term

$$\mathcal{T} = m \int dx \left(k_I \dot{X}^I + u_I X^{'I} \right) \ , \tag{3.2}$$

which is simply a mixture of the Noether charge associated to the symmetry generated by k^I and a topological charge arising from the potential. From (3.1) we can read off a simple set of Bogomoln'yi equations to be

$$\dot{X}^I = \frac{1}{2} m k^I \ , \qquad X'^I = \frac{1}{2} m u^I \ . \tag{3.3}$$

In the dual model the equations would be simply written in terms of the hatted fields. One can check from the above formulae that $\mathcal{T}$ and the $I = i$ components of (3.3) are unchanged by T-duality. However the $I = 0$ components of (3.3) in the original and T-dual model are

$$\begin{aligned} \dot{X}^0 &= \frac{1}{2} m \ , \qquad & X'^0 &= -\frac{1}{2} m g^{0i} b_{0i} \ , \\ \dot{\hat{X}}^0 &= 0 \ , \qquad & \hat{X}'^0 &= \frac{1}{2} m (g_{00} + g^{ij} b_{0i} b_{0j}) \ . \end{aligned} \tag{3.4}$$

Thus the BPS states still exist in the T-dual model, but they take on a different form. In particular the momentum modes have disappeared in the dual model. In

the example below we will see that they have in fact turned into winding modes, in analogy to the interchange of the string winding and momentum modes under T-duality.

To conclude let us consider an example given by the $(4,4)$ supersymmetric Q-kink sigma model [9] with vanishing torsion and metric

$$ds^2 = H^{-1}(dX^0 + \omega_i dX^i)^2 + H\delta_{ij}dX^i dX^j \ , \tag{3.5}$$

where X^0 is periodic with period 4π, $i = 1,2,3$ and ω_i is defined by $\frac{1}{2}\epsilon_{ijk}\partial_j\omega_k = \partial_i H$. Here H is the harmonic function

$$H = \delta + \sum_{n=1}^{N} \frac{1}{\mid X^i - Y^i_n \mid} \ , \tag{3.6}$$

where $\delta = 0,1$ for an ALE or multi-Taub-Nut space respectively, the later case describing a KK monopole. There is a unique potential which can be added preserving $(4,4)$ supersymmetry and it is given as the length of the Killing vector $k = \partial/\partial X^0$;

$$V = \frac{1}{4}m^2 H^{-1} \ . \tag{3.7}$$

Thus the supersymmetric vacua of this theory are the located at the zeros of H^{-1}, i.e. at the centres Y^i_n of the metric.

If we now dualise along X^0 we obtain

$$\begin{aligned} d\hat{s}^2 &= H\left((d\hat{X}^0)^2 + \delta_{ij}d\hat{X}^i d\hat{X}^j\right) \ , \\ \hat{b}_{0i} &= \omega_i \ , \end{aligned} \tag{3.8}$$

which also admits $(4,4)$ supersymmetry [10] and describes a solitonic 5-brane. As we showed above the potential (3.7) remains the same, although now it is given by the length of the non-trivial 1-form $\hat{u}_I = (1,0,0,0)$. Furthermore one can check that $(4,4)$ supersymmetry is preserved by the potential (3.7) in the dual model.

Let us now compare the BPS soliton solutions of these two models. Because our analysis of these states above assumed only $(1,1)$ supersymmetry it overlooks the quarternionic structure underlying these models. Let us then briefly recall the analysis of [9] for the model (3.5). Introducing a triplet $\mathbf{I}^I{}_J$ of complex structures one can deduce the richer bound

$$\begin{aligned} E = \int dx \Big\{ & g_{IJ}(X'^I - \frac{1}{2}m(\mathbf{n}\cdot\mathbf{I})^I{}_K k^K)(X'^J - \frac{1}{2}m(\mathbf{n}\cdot\mathbf{I})^J{}_L k^L) \\ & + g_{IJ}(\dot{X}^I - \frac{1}{2}mn_0 k^I)(\dot{X}^J - \frac{1}{2}mn_0 k^J)\Big\} + m(n_0 Q_0 + \mathbf{n}\cdot\mathbf{Q}) \ , \end{aligned} \tag{3.9}$$

where $(n_0, \mathbf{n})$ is a unit vector and

$$Q_0 = \int dx \dot{X}^I k_I \ , \qquad \mathbf{Q} = \int dx X'^I k^J \mathbf{I}_{IJ} \ . \tag{3.10}$$

are the Noether charge and three topological charges. By choosing $(n_0, \mathbf{n})$ parallel to $(Q_0, \mathbf{Q})$ and setting the squared terms in (3.9) to zero we obtain the Bogomoln'yi equations

$$\dot{X}^0 = \frac{1}{2}mn_0 \ , \qquad \dot{X}^i = 0 \ , \qquad X'^0 = 0 \ , \qquad X'^i = \frac{1}{2}mn^i H^{-1} \ , \tag{3.11}$$

with the energy of the given by $m\sqrt{Q_0^2 + \mathbf{Q}\cdot\mathbf{Q}}$. Thus one finds the Q-kink BPS solutions [9]*

$$\begin{aligned} X^0 &= X^0_0 + \frac{1}{2}mn_0 t \ , \\ X^i &= \frac{1}{2}(Y^i_1 + Y^i_2) + \frac{1}{2}(Y^i_1 - Y^i_2)\tanh\left(\frac{m \mid \mathbf{n} \mid}{16}(x - x_0)\right) \ , \end{aligned} \tag{3.12}$$

where X^0_0 and x_0 are arbitrary constants and Y^i_1 and Y^i_2 are any two distinct vacua of the potential (3.7).

$\star$ These solutions are only for the case $\delta = 0$. For $\delta = 1$ there also exist solutions but they can not be expressed in a closed form.

Now consider the T-dual theory. The analysis follows along similar line except that that now we must use $\hat{u}^I$ instead of k^I (the complex structures are also different in the two models [6]). In this case the charges are given by (cf. (3.2))

$$\hat{Q}_0 = \int dx \hat{X}'^I \hat{u}_I \ , \qquad \hat{\mathbf{Q}} = \int dx \hat{X}'^I \hat{u}^J \hat{\mathbf{I}}_{IJ} \ . \tag{3.13}$$

Note that now all the charges are topological. In order to obtain the correct Bogomoln'yi bound we may write down a similar expression to (3.9)

$$\begin{aligned} E = \int dx \Big\{ & \hat{g}_{IJ}(\hat{X}'^I - \frac{1}{2}m(n_0\hat{u}^I + \mathbf{n}\cdot\hat{\mathbf{I}}^I{}_K\hat{u}^K))(\hat{X}'^J - \frac{1}{2}m(n_0\hat{u}^J + \mathbf{n}\cdot\hat{\mathbf{I}}^J{}_L\hat{u}^L)) \\ & + \hat{g}_{IJ}\dot{\hat{X}}^I\dot{\hat{X}}^J \Big\} + m(n_0\hat{Q}_0 + \mathbf{n}\cdot\hat{\mathbf{Q}}) \ . \end{aligned} \tag{3.14}$$

We therefore arrive at the dual Bogomoln'yi equations

$$\dot{\hat{X}}^I = 0 \ , \qquad \hat{X}'^0 = \frac{1}{2}mn_0H^{-1} \ , \qquad \hat{X}'^i = \frac{1}{2}mn^iH^{-1} \ , \tag{3.15}$$

The energy of these solutions given by $m\sqrt{\hat{Q}_0^2 + \hat{\mathbf{Q}}\cdot\hat{\mathbf{Q}}}$ and the BPS states of the dual theory are

$$\begin{aligned} \hat{X}^0 &= \hat{X}^0_0 + \frac{1}{2}n_0\frac{(\mid Y_1\mid^2 - \mid Y_2\mid^2)}{\mid Y_1 - Y_2\mid} + \frac{1}{2}n_0 \mid Y_1 - Y_2\mid \tanh\left(\frac{m\mid\mathbf{n}\mid}{16}(x-x_0)\right) \ , \\ \hat{X}^i &= \frac{1}{2}(Y_1^i + Y_2^i) + \frac{1}{2}(Y_1^i - Y_2^i)\tanh\left(\frac{m\mid\mathbf{n}\mid}{16}(x-x_0)\right) \ . \end{aligned} \tag{3.16}$$

Thus the Q-kink momentum modes have become wrapping modes around the compact dimension. It is pleasing to see that the notion of T-duality exchanging momentum and winding modes about the Killing direction extends to the BPS states of the worldsheet. Note that the charges, and the hence the masses, for the these states are the same before and after T-duality as one would expect.

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